ON THE TEACHING OF THE REPRESENTATION OF COMPLEX NUMBERS IN THE ARGAND DIAGRAM

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Abstract

This paper reports a conceptualisation of the teaching of the representation of complex numbers in the Argand.Existing literature on different teaching approaches of complex numbers was reviewed and the "new" method proposed here incorporates the strengths of the various traditional approaches to form a feasible lesson. A lesson outline and the lesson material with explanatory notes are included.

Introduction

In the University of Cambridge (Advanced Level) mathematics curriculum, representation of complex numbers in the Argand diagram is taught within the topic of complex numbers at thepre-universitylevel.

The geometric representation using the Argand diagram is essential for overall understanding of the topic of complex numbers (Leitzel, 1989). Without geometric representation, complex numbers would be "operated upon in a purely symbolic and algorithmic mode" (Panaoura, Elia, Gagatsis&Giatilis, 2006, p. 684).Sfard (1991) asserts that the Argand diagram provides a visual representation, which helps studentsrectify the concept of complex numbers and progress from operational understanding to structural understanding.

Learning Difficulties, Errors and Misconceptions

An understanding of the issues students encounter when learning the representation of complex numbers in the Argand diagram is essential in the planning of instructional strategies to overcome these difficulties.

Learning difficulties

Students are confused between the algebraic representation of complex numbers and the geometric representation of complex numbers. An understanding of one representation does not necessarily imply an understanding of the other representation.Panaoura et al. (2006) found that some students "considered the geometric and algebraic representation as two different and autonomous mathematical objects and not as two means of representing the same concept" (p. 701). Since the ability to identify and represent a concept in different ways is vital for deeper understanding, this compartmentalisationsuggests a fragmental understanding of complex numbers.

Even when students are able to view the algebraic and geometric representations as just two different ways of representing complex numbers, they are still much less comfortable carrying outoperations on complex numbers in the Argand diagram than by direct calculation.

In a study by Connor, Rasmussen, Zandieh and Smith (2007) on ten students enrolled in a course for prospective secondary school teachers, all ten students were found to be comfortable performing basic complex arithmetic such as addition and multiplication by direct calculation at the start of the course. However, only one student was comfortable carrying out the same operations using the Argand diagram, and this only increased to six upon the completion of the course. The results point to a lack of familiarity with the geometric representation of complex numbers.

Anecdotal accounts from mathematics classrooms in Singapore reveal that pre-university students question the usefulness of the geometric representation of complex numbers in the Argand Diagram (hereafter abbreviated as*this sub-topic*),as the mathematics content appears to be completely detached from reality.

Errors

According to Swan (2001), errors are mistakes due to “lapses in concentration, hasty reasoning, memory overload or a failure to notice salient features of a situation” (p. 147), and these can be attributed to a lack of self-regulation of learning.

Both graphs and Argand diagrams are pictorial representations, hence some errors in the learning of graphs also occur in the learning of Argand diagrams. Firstly, students neglect to label the real and imaginary axes. Secondly,students choose inappropriate scales. They insist on using the same scale for both real and imaginary axes even when it is advisable to use different scales for certain questions. In addition, they do not maintain the same scale across positive and negative sections of an axis in the Argand diagram. Thirdly, students incorrectly plot points, especially those with non-integer coordinates, because they read off their chosen scales erroneously (Chua & Ng, 2009).

Misconceptions

Misconceptions arealternative interpretations of concepts held by students, and these are contrary to what has been taught (Swan, 2001).

A common misconception in this sub-topicoccurs when students are taught to interpretcomplex numbers as vectors in the Argand diagram. For a complex number $z = x + iy$represented by a position vector $\overrightarrow{OP}$ whose starting point is the origin O and ending point is P with coordinates (x, y), students may write $z = \overrightarrow{OP}$ if they do not realize that representing the construct of complex numbers by another construct of vectors does not mean that the two constructs are equivalent (Evans, 2006).

Possible Teaching Approaches

A review of literature on the teaching of complex numbers yields several approaches for the teaching of this sub-topic. Each approach has its advantages and disadvantages, and it is worth noting that the approaches are not necessarily incompatible with one another.

Traditional approach

Complex numbers are introduced using the need to solve polynomial equations as a motivation.The idea of representing complex numbers by points on the Argand diagram is then presented as a fairly obvious and natural step (Driver &Tarran, 1989). The geometric effects of operations on complex numbers as presented in the teaching notes of the Singapore schools are usually stated as facts to be memorised:

- Addition and subtraction of complex numbers correspond to the parallelogram law of vector addition and subtraction.
- If a point P represents a complex number z, then the point representing iz is obtained by rotating OP $90°$ anti-clockwise about the origin.

Driver and Tarran (1989)[1] term this the "expediency approach" (p. 122) because it is the most direct approach taking into account examination requirements. Another advantage is that it closely mirrors the development of complex numbers throughout mathematical history. However, many teachers reduce rich relationships within the topic to a few simple statements, depriving students of opportunities to think (Fung, Siu, Wong & Wong, 1998).

Complex numbers as vectors

In addition to representation by points in the Argand diagram, complex numbers can also be represented by vectors in the Argand diagram. This offers a convenient geometric model to explain the additive structure of complex numbers, as well as to interpret the multiplicative structure of complex numbers in terms of compositions of scalings and rotations (Evans, 2006). Indeed, Hahn (1994) recommends representing complex numbers by points or vectors in the Argand diagram depending on which identification is more useful for the given context (pp. 1 – 54). This allows material from vectors to be used to support the development of concepts in complex numbers. Specifically, "techniques and operations used in vector problems can be applied equally well to complex number analysis" (Bostock& Chandler, 1981, p. 543). Hence, connections can be made between two seemingly disparate topics, enhancing students' overall appreciation for the subject (Evans, 2006).However, careful sequencing of topics is required to take advantage of the links, and teachers mustalso anticipate students' misconception intreating complex numbers and vectors as equivalent constructs.

Using Information and Communication Technology (ICT)

Mathematics educators are increasingly leveraging on ICT today to develop students' interest in the subject and enhance their learning experience (Wong, 2009).Evans and Oldknow (1996) developed a method of teaching geometric effects of operations on complex numbers using graphic calculators. In this approach, a set of complex numbers, whose points define a polygon when plotted, is stored as a list. An operation, such as multiplication by i,is then applied to every element in the initial list with the result being stored in a new list. Both the initial and transformed polygons are then plotted, leading students to conclude that multiplication by i corresponds to a $90°$ anti-clockwise rotation about the origin.

Here, the graphic calculator is employed in the tool mode where students use technology to learn and apply mathematics. Wong (2009) advocates this mode because it fosters conceptual understanding. Moreover, this enables students to learn through hands-on experience, instead ofbeing presented with information in a didactic manner.

A drawback of this approach is that procedures for carrying out such transformations in graphic calculators are rather involved, necessitating a certain degree of familiarity with lists and parametric mode plotting, and this may distract students' attention from key concepts.

[1] Driver and Tarran (1989) suggest four other teaching approaches in their article – complex numbers as operators, complex numbers as ordered pairs of real numbers which satisfy axioms of a group and a field, complex numbers as polynomial residues to modulus $x^2 + 1$, and through spiral similarities of transformations in a plane. However, these are beyond the scope of the H2 mathematics curriculum and hence are not discussed in this paper.

Furthermore, this approach should not be used in isolation because it neitherdevelops the concept of the Argand diagram explicitly nor provides opportunities for students to practise representing complex numbers on the Argand diagram by hand.

Mathematical applets on the Argand diagram are readily available on the Internet. These applications are easy to use, but teachers must exercise care to ensure that the applets are meaningful and mathematically correct. For example, the applet atWaldo's Maths Pages (Barrow, 2001) demonstratesall key concepts in this sub-topic; students caneasily manipulate complex numbers in an intuitive-experimental manner.

Using a story

Another resource available on the Internet is "John and Betty's Journey into Complex Numbers" (Bower, n.d.). This electronic book is a story about two children, John and Betty, who solve a series of problems designed to introduce complex numbers and related concepts such as the Argand diagram.This resource corresponds to the story mode in the multi-modal approach put forward by Wong (1999) to deepen mathematical understanding. It motivates learning in a way that is enjoyable and intuitive for students. However, this approach is insufficient if used in isolation, and should be adopted instead as a supplementto other selected approaches.

Using fractals

Forster (1997) designed a series of lessons on complex numbers for Australian Year 12 students using fractals in an anchored instruction approach. Posters of fractals were used to decorate the classroom before the topic of complex numbers was introduced. Students were then told that the concepts taught in subsequent lessons would help them understand how fractals were generated and enable them to create fractals on their own. After learning concepts such as the Argand diagram, addition of complex numbers and the modulus, students proceeded to calculate, draw and color fractals by hand before generating fractals on a computer using a fractal generator available on the Internet.

Fractals occur widely in nature and have direct applications in fields as diverse as music, architecture and economics (Mandelbrot & Frame, 2002). As it is important to relate mathematics to real-life applications, this approach may be useful in motivating students.

Discovery approach

In this approach, students are guided to discover knowledge on their own by working on various activities (Bruner, 1974). This was carried out in teaching the geometric effect of adding two complex numbers through the following series of questions (The Open University, 1981, p. 12):

(i) (a) Mark 1, i and $1 + i$ on an Argand diagram.
(b) Add $2 + i$ to each of the above complex numbers, and mark the new numbers on the *same* diagram.

(ii) What effect, geometrically, does adding $2 + i$ have on the position of a complex number on the Argand diagram?

(iii) What is the geometric interpretation of adding $a + bi$ to a complex number?

The advantage of this approach is that students have a greater sense of ownership in their learning. Moreover, this avoids procedural understanding. However, students may discover the wrong mathematics or become frustrated at not knowing what to discover.

Comparing the geometry of the real number line and the Argand diagram

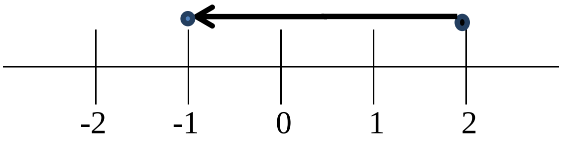
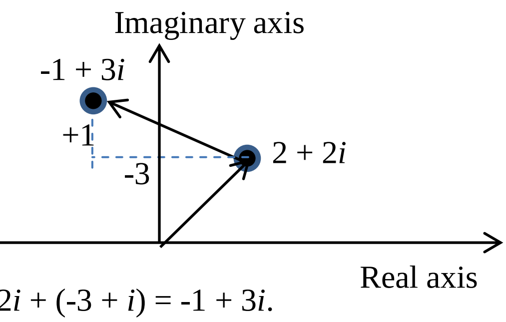
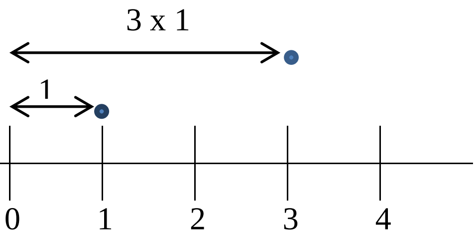
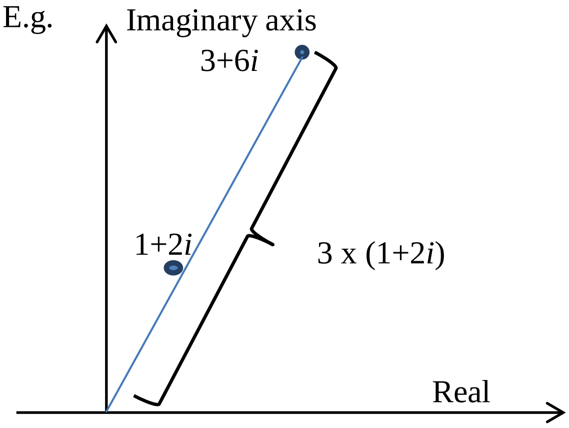
Students first investigate the geometry of the real number line (Dolan et al., 1991, p. 2):

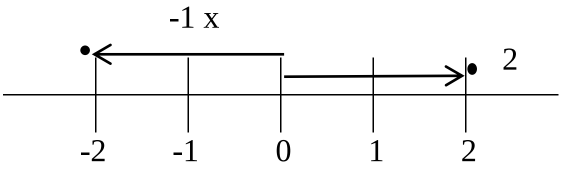
- Addition on the number line is vectorial
- The effect of multiplying a number by a positive constant, c, is a stretch from the origin, scale factor c
- The effect of multiplying by -1 is an anti-clockwise rotation of 180° about the origin

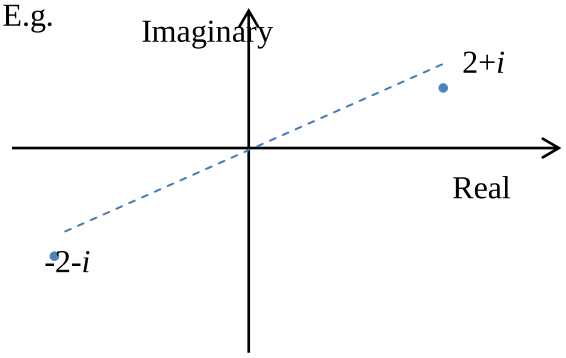
After the Argand diagram and geometrical effects of operations on complex numbers have been taught, a comparison between the geometry of the real number line and the geometry of the Argand diagram can then be carried out as shown below in Table 1:

Table 1

A Comparison of the Geometry of the Real Number Line and the Argand Diagram

Real number line	Argand diagram
Addition on the number line is vectorial in one dimension: E.g.  2 + (-3) = -1.	Addition in the Argand diagram is vectorial in two dimensions: E.g.  2 + 2i + (-3 + i) = -1 + 3i.
The effect of multiplying a number by a positive constant, c, is a stretch from the origin, scale factor c. E.g.  3 (1) = 3	The effect of multiplying a complex number by a positive real number, c, is a stretch from the origin, scale factor c. E.g.  3 (1 + 2i) = 3 + 6i.

<table>
<tr>
<td>The effect of multiplying by −1 is an anti-clockwise rotation of 180° about the origin
E.g.


-1 x 2 = -2 (by reflecting about the origin 0)</td>
<td>The effect of multiplying a complex number by −1 is an anti-clockwise rotation of 180° about the origin.
E.g.


-1 x (2+*i*) = -2 – *I* (by reflecting the original point about the origin 0)

Note that the geometrical effect of multiplying a complex number by *i*, has no counterpart in the real number system.</td>
</tr>
</table>

This approach creates conceptual connections across various mathematical domains such as real numbers, complex numbers and vectors, making this sub-topic more meaningful for students (Lakoff & Núñez, 2000, p. 422). However, referring to multiple domains may result in cognitive overload among weaker students.

Selected Teaching Approaches and Rationale

According to Driver and Tarran (1989), it is common to use only one approach in the teaching of the representation of complex numbers in the Argand diagram, but this may not be particularly beneficial for students. On the other hand, different approaches can "help to forge the links which are necessary for a true conceptual understanding" (p. 22) of this sub-topic.

Based on the teaching approaches presented earlier, the author has selected a combination of the aboveapproaches for developing one lesson on teaching the representation of complex numbers on the Argand Diagram:

- Discovery approach
- Using fractals (as a real-life application of the Argand diagram)
- Using ICT (in the form of a mathematical applet)
- Complex numbers as vectors

The traditional approach (Section 3.1) fosters instrumental understanding, which Skempcalls "rules without reasons" (1976, p. 20). While instrumental mathematics produces more immediate results, it is also easier to forget. On the other hand, relational mathematics, which involves knowing both what to do and why, is more lasting. It also has the added advantage of being more adaptable to new problems.

In the context of this sub-topic, constructivism would imply allowing students to formulate the geometric interpretations of operations on complex numbers on their own. Moreover, the mathematical content in this sub-topic is not way beyond students' abilities, and pre-university students are at the formal operational stage according to Piaget's Cognitive Development Theory: they are able to think in logical and abstract ways, make conjectures and test hypotheses (Santrock, 2009). Hence, it is appropriate to employ constructivism in the teaching of this sub-topic.

From a constructivist perspective, Bruner's discovery approach (1974) is an excellent teaching and learning strategy. Students can be engaged through a worksheet so that they can discover conceptson their own. The provision of probing questions and hints in the worksheet, as well as the division of problems into smaller steps, also allows for teaching in what Vygotsky termed as the zone of proximal development, which is the stage where students cannot master a task on their own but can do so with appropriate guidance and assistance from the teacher or more capable peers (Santrock, 2009).

Worksheet (part of which is described in Section 5) can be designed to develop the geometric representation of complex numbers so that students will not view the algebraic and geometric representations as separate. Sufficient questions should be provided so that students are comfortable carrying out operations on complex numbers in the Argand diagram.In addition, the inclusion of information about fractals helps students realize the applications of the mathematical content in reality.

Misconceptions are also dealt with in the worksheet through thinking activities, which consist of carefully engineered questions to induce cognitive conflict that is subsequently resolved through a whole-class discussion (Swan, 2001). Errors are detected and pointed outas the teacher circulates around the classroom while students attempt the worksheet.

The mathematical applet complements the constructivist approach by allowing students to test their conjectures before formulating the geometric interpretations of operations on complex numbers. In addition, according to information-processing theories, connecting the topic of complex numbers with vectors helps in students' memory encoding process, and enhances storage and retrieval (Santrock, 2009).

Hence, the selected approaches will help students better learn concepts astheyadhere to the three principles of how students learn mathematics (National Research Council, 2005):

- Building on prior knowledge
- Building conceptual understanding, procedural fluency and connected knowledge
- Building resourceful, self-regulating problem solvers

Lesson Outline and Sample Section of Worksheet

This section provides the detail of a lesson on Complex Numbers, on the representation of complex numbers on an Argand Diagram.

Topic: Complex Numbers
Section: Representation of Complex Numbers on an Argand Diagram
Objectives of this lesson: Students should be able to
(a) represent complex numbers expressed in Cartesian form by points in the Arganddiagram;

(b) interpret the terms “real part” and “imaginary part” of a complex number geometrically;

(c) describe the geometrical effect of arithmetic procedures involving complex numbers geometrically: conjugating a complex number, adding and subtracting of two numbers and multiplying a complex number by *i*.

The proposed duration of this lesson is one hour. The detail of the lesson activity is presented below.

The teacher first captures the attention of students at the start of the tutorialby telling a short but interesting story about Jean-Robert Argand. Using the story, the teacher leads students to the concept of the Argand diagram as a means of representing complex numbers geometrically. Students are then guided to discover the geometric effects of operations on complex numbers by working on a worksheet. The teacher concludes the tutorial with a brief recap on the important concepts covered during the lesson.

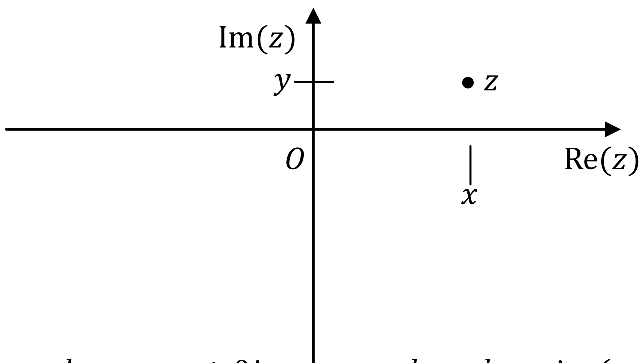
Do you know Jean-Robert Argand?

Jean-Robert Argand was born in Geneva, Switzerland in 1768. After moving to Paris, he became the manager of a bookstore. Despite lacking formal training in mathematics, Argand pursued mathematics as a hobby in his free time. In 1806, Argand wrote an essay on complex numbers which was published by a small printing company. Since he intended to give away copies of his essay to friends, who would have known who the author was,Argand did not even include his name on the title page. However, this humble work actually managed to solve a problem that had stumped many talented mathematicians for over 200 years!

After stimulating students’ interest based with a “humble beginning”, and which students might actually want to know what Argand has written, the teacher will next begin the proper introduction of the Argand Diagram.

Part 1 of the Lesson. The Argand diagram

Real numbers can be represented geometrically by points on a real number line. However, it is not possible to do so for complex numbers. How can complex numbers be represented geometrically? In Argand’s 1806 essay, he suggested that since a complex number $z = x + iy$ could be represented by the ordered pair (x, y) where $x, y \in \mathbb{R}$, it could be plotted as a point on a plane with cartesian co-ordinates (x, y):



- *Since a real number $x = x + 0i$ corresponds to the point $(x, 0)$ on the horizontal axis, the horizontal axis is called the* ________________________________.
- *Since an imaginary number $iy = 0 + iy$ corresponds to the point $(0, y)$ on the vertical axis, the vertical axis is called the* ________________________________.

The above simply examples of representing a real number on the Argand Diagram easily leads students to realise that the horizontal axis, the real axis, is actually the Real Number Line that they have learnt in the lower secondary school mathematics curriculum.What is "new" now is the vertical imaginary axis.

Question 1.*Represent the following complex numbers as points on a single Argand diagram and label them accordingly:*

(i) $z_1 = -5 + 4i$ *(ii)* $z_2 = 6$

Part two of this lesson continues with the more delicate representation of arithmetical operations of complex numbers, in particular, conjugation.

Part 2 of the Lesson. Representing Complex Conjugates
Recall that if $z = x + iy$, *then* $z^* = x - iy$.

Question 2. *Suppose we have* $z_1 = 0.5 + i$ *and* $z_2 = -2 - 5i$.

(i) *Find* z_1^* *and* z_2^* *and represent all 4 complex numbers on a single Argand diagram.*
(ii) *What do you notice about the relative positions of the complex numbers and their conjugates?*
(iii) *Visit http://www.waldomaths.com/Complex1N.jsp. Select the "Conjugate" option in the mathematical applet and move* z_1*throughout all four quadrants of the Argand diagram to test your conjecture.*
(iv) *What can you conclude?*

Note that part two of this lesson involves also the use of technological tools to explore the relationship between complex numbers and their conjugates. It appears that this section will take more time. However, it should be noted that once students have mastered the use of the tools, the next lesson on further geometrical representation of other operations of complex numbers (addition, subtraction and multiplication by *i*) will take a shorter time to complete. One might perhaps feel that much time is spent on this lesson on rather rudimentary material.However, it should also be noted that once students are enthused and want to learn more about this subtopic, much time could be saved in subsequent lessons. Learning of this topic has become meaningful to the students. Furthermore, they have acquired autonomy in their learning process with the use of technological tools, deductive reasoning and appreciation of historical origin of the related mathematical concepts.

Pedagogical Considerations

The representation of complex numbers in the Argand diagram is only taught after sub-topics such asroots of a polynomial equation to give teachers time to develop the notion that complex numbers are the result of extending the number system from real numbers. By first learning the operations on complex numbers algebraically, students will also appreciate that the algebraic and geometric representations are simply two different ways of representing the concept of complex numbers. Furthermore, this sub-topic is taught before the trigonometric and exponential forms of complex numbers because the latter requires concepts of modulus and argument, which can be better taught using the Arganddiagram (Driver & Tarran, 1989).

Pre-universities use the lecture-tutorial system in their delivery of lessons. While lectures are efficient in the exposition of content to large numbers of students, they are not learner-centred and active student participation is limited (Brown & Atkins, 1988). In contrast, tutorials allow for greater teacher-student and student-student interaction, as well as the provision of immediate feedback from teachers, and are more suited to a constructivist teaching approach. Hence, the proposed lesson is conceptualised as a tutorial, complementing other lectures and tutorials on the topic of complex numbers.

Mathematical history is used to arouse students' curiosity. According to Gagné (1985), it is important to gain the attention of students at the start of a lesson for successful teaching and learning to take place (p. 243 – 258). However, the judicious use of mathematical history is important as students may be intimidated by a presentation of cold historical facts. Instead, astory about Jean-Robert Argand is recounted in the proposed lesson. Excellent sources on the history of complex numbers include Nahin (1998) and Burton (2007).

Motivating the Argand diagram through the real number line is a way of extending from the known to the unknown. This enables students to construct understanding by using and adapting their schemas through the Piagetian concepts of assimilation, where students incorporate new information into existing knowledge, and accommodation, where students modify schemas to incorporate new information and experiences (Santrock, 2009).

Scaffolding is requiredin the initial stages of discovery to help students attain the upper limits of their zone of proximal development (Yeo, Hon, & Cheng, 2006). This is provided in the worksheet in the form of probing questions and hints. As student competence increases, scaffolding is reduced and eventually removed.

In addition, ample questions should be provided in the worksheet to help students internalize new concepts. According to Thorndike (1913), practice strengthens connections. In accordance with Thorndike's Law of Recency, questions are also posed after every new concept is introduced, instead of being aggregated at the conclusion of the worksheet when all the concepts have been taught (pp. 170–194). Following Dienes' Principle of Mathematical Variability, questions involve complex numbers in different quadrants of the Argand diagram, so that students are able to identify defining and non-defining attributes.

The discussion of the operations on complex numbers through algebraic representation, calculations involving specific complex numbers, the Argand diagram and the formulation of geometric interpretations correspond to the symbol, number, diagram and word modes respectively in Wong's multi-modal approach (1999). This builds flexible connections among multiple modes of representation, and since students have different learning styles, they can "learn mathematics using one or more thinking modes that suit them best" (p. 9).

Conclusion

In this paper, a conceptualisation of the teaching of the representation of complex numbers in the Argand diagram is presented. Possible teaching approaches have been considered and the rationale has been provided for the selected approaches. Pedagogical considerations behind the proposed lesson have also been discussed.The detailed material for the entire one-hour lessonhas been included in Section 5 for reference.

References

Barrow, R. F. (2001). Introduction to Complex Numbers [Java Applet]. Retrieved October 17, 2009, fromhttp://www.waldomaths.com/Complex1N.jsp

Bostock, L., & Chandler, S. (1981). *Mathematics: The core course for A-level*. Cheltenham, UK: Stanley Thornes.

Bower, M. (n.d.). *John and Betty's journey into complex numbers: A children's book for senior school mathematics students*. Retrieved October 17, 2009, from http://mathforum.org/johnandbetty/

Brown, G., & Atkins, M. (1988).*Effective teaching in higher education*. London: Routledge.

Bruner, J. S. (1974). Some elements of discovery. In J. S. Bruner (Ed.), *The relevance of education* (pp. 84-97). Middlesex, UK: Penguin Education.

Burton, D. M. (2007). *The history of mathematics: An introduction* (6thed.). New York: McGraw-Hill.

Chua, B. L., & Ng, K. E. D. (2009). Teaching of Graphs. In P. Y. Lee & N. H. Lee (Eds.), *Teaching Secondary School Mathematics: A Resource Book* (2nd ed., pp. 51-85). Singapore: McGraw-Hill.

Connor, M. E., Rasmussen, C., Zandieh, M., & Smith, M. (2007). *Mathematical knowledge for teaching: The case of complex numbers*. Paper presented at the 2007 Conference on Research in Undergraduate Mathematics Education. Retrieved October 2, 2009, from http://www.rume.org/crume2007/papers/conner-rasmussen-zandieh-smith.pdf

Dolan, S., Haydock, R., Hunt, B., Lyons, L., Price, P., Roder, P., & Williamson, K. (1991). *Complex numbers: The school mathematics project*. Cambridge, UK: Cambridge University Press.

Driver, D. A., & Tarran, D. S. G. (1989). Five approaches to the teaching of complex numbers. *Teaching mathematics and its applications, 8*(3), 122-127.

Evans, L. (2006). *Complex Numbers and Vectors*.Camberwell, Australia: ACER Press.

Evans, W., & Oldknow, A. (1996).Micromaths: Approaching complex numbers through transformations. *Teaching Mathematics and its Applications, 15*(2), 91-93.

Forster, P. (1997). Using fractals to teach complex numbers with a constructivist approach.*Australian Senior Mathematics Journal, 11*(2), 14-22.

Fung, C. I., Siu, M. K., Wong, K. M., & Wong, N. Y. (1998). A dialogue on the teaching of complex numbers and beyond.*Mathematics Teaching, 164*, 26-31.

Gagné, R. M. (1985). *The conditions of learning and theory of instruction* (4thed.). New York: CBS College Publishing.

Hahn, L. S. (1994). *Complex Numbers & Geometry*. Washington, DC: Mathematical Association of America.

Lakoff, G., & Núñez, R. E. (2000).*Where Mathematics Comes From: How The Embodied Mind Brings Mathematics Into Being*. New York: Perseus Books.

Leitzel, J. R. (1989). Critical considerations for the future of algebra instruction. In S. Wagner & C. Kieran (Eds.), *Research Issues in the Learning and Teaching of Algebra* (Vol. 4, pp. 25-32). Reston, VA: National Council of Teachers of Mathematics.

Mandelbrot, B. B., & Frame, M. (2002). Some reasons for the effectiveness of fractals in mathematics education. In M. Frame & B. B. Mandelbrot (Eds.), *Fractals, graphics,* *and mathematics education* (pp. 3-10). Washington, DC: Mathematical Association of America.

Nahin, P. J. (1998). *An imaginary tale: The story of* $\sqrt{-1}$. Princeton, NJ: Princeton University Press.

National Research Council. (2005). Howstudents learn: Mathematics in the classroom. In M.S. Donovan & J.D. Bransford (Eds.), *How People Learn: A Targeted Report for Teachers* (pp. 217-393). Washington, DC: The National Academies Press.

Panaoura, A., Elia, I., Gagatsis, A., & Giatilis, P. (2006). Geometric and algebraic approaches in the concept of complex numbers.*International Journal of Mathematical Education in Science and Technology, 37*(6), 681-706. Retrieved September 27, 2009, from ERIC.

Santrock, J. W. (2009). *Educational Psychology* (4thed.). New York: McGraw-Hill.

Sfard, A. (1991). On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics, 22*(1), 1-36. Retrieved October 10, 2009, from JSTOR.

Skemp, R. R. (1976). Relational understanding and instrumental understanding.*Mathematics Teaching*, *77*, 20-26.

Swan, M. (2001). Dealing with misconceptions in mathematics.In P. Gates (Ed.), *Issues in Mathematics Teaching* (pp. 147-165). London: RoutledgeFalmer.

The Open University. (1981).*Complex Numbers*. Milton Keynes, UK: Author.

Thorndike, E. L. (1913). *Educational Psychology* (Vol. 1). New York: Teachers College, Columbia University.

Wong, K. Y. (1999). *Multi-modal approach of teaching mathematics in a technological age*. Paper presented at the 8th Southeast Asian Conference on Mathematics Education (SEACME-8). Retrieved October 13, 2009, from http://math.nie.edu.sg/kywong/Multi-modal%20SEACME%20paper%20with%20added%20diagram.pdf

Wong, K. Y. (2009). ICT and mathematics education. In P. Y. Lee & N. H. Lee (Eds.), *Teaching Secondary School Mathematics: A Resource Book* (2nd ed., pp. 357-367). Singapore: McGraw-Hill.

Yeo, B. W., Hon, S. F., & Cheng, T. G. H. (2006). *Engaged learning in mathematics.* Paper presented at the 2006 Educational Research Association of Singapore (ERAS) Conference. Retrieved October 13, 2009, from http://math.nie.edu.sg/bwjyeo/publication/ERAS2006Paper_EngagedLearning1.pdf